STABILITY OF PERIODIC SOLUTIONS OF QUASILINEAR AUTONOMOUS SYSTEMS WITH ONE DEGREE OF FREEDOM

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Consideration is given to the stability of periodic solutions of quasilinear autonomous systems for which the equation of fundamental amplitudes has multiple roots. In this case the periodic solutions can be represented in the form of series in integral, as well as fractional, powers of a small parameter.

1. Let us consider the oscillatory system

$$\frac{d^2x}{dt^2} + p^2x = \mu f\left(x, \frac{dx}{dt}, \mu\right) \tag{1.1}$$

The function $f(x, \dot{x}, \mu)$ is assumed to be analytic in its arguments in some region of their variation; the parameter μ is small and positive. When $\mu = 0$, a solution of the generating equation is

$$x_0(t) = A_0 \cos pt \tag{1.2}$$

The initial conditions for the system (1.1) can be taken (in view of the autonomous nature of our system) as

$$x(0) = A_0 + \beta, \qquad \dot{x}(0) = 0 \tag{1.3}$$

where β is a function of μ which vanishes when $\mu = 0$.

The solution of the equation (1.1) can be expressed in the form [1]

$$x(t, A_0 + \beta, \mu) = (A_0 + \beta)\cos pt + \sum_{n=1}^{\infty}\left[C_n(t) + \frac{\partial C_n}{\partial A_0}\beta + \frac{1}{2}\frac{\partial^2 C_n}{\partial A_0^2}\beta^2 + \ldots\right]\mu^n \tag{1.4}$$

Here the function $C_n(t)$ is determined by the formulas (1.5)

$$C_n(t) = \frac{1}{p}\int_0^t H_n(t_1)\sin p(t - t_1)\,dt_1, \qquad H_n(t) = \frac{1}{(n-1)!}\left(\frac{d^{n-1}f}{d\mu^{n-1}}\right)_{\beta=\mu=0}$$

The oscillation period of the autonomous system depends on the parameter μ

$$T = T_0 + \alpha, \qquad T_0 = 2\pi / p \tag{1.6}$$

where α is a function of μ that vanishes when $\mu = 0$. From the condition of periodicity of the derivative $\dot{x}(t)$ one can find the function $\alpha = \alpha(A_0 + \beta, \mu)$. Substituting this function into the condition for periodicity of $x(t)$, we obtain the equation for the determination of the fundamental amplitudes A_0

$$M_1 = C_1(T_0) = -\frac{1}{p}\int_0^{T_0} f(x_0, \dot{x}_0, 0)\sin pt\,dt = 0 \tag{1.7}$$

and also the equation that determines β as an implicit function of μ

$$\Phi(\beta, \mu) = \frac{\partial C_1}{\partial A_0}\beta + M_2\mu + \frac{1}{2}\frac{\partial^2 C_1}{\partial A_0^2}\beta^2 + \frac{\partial M_2}{\partial A_0}\beta\mu + M_3\mu^2 + $$
$$+ \frac{1}{6}\frac{\partial^3 C_1}{\partial A_0^3}\beta^3 + \frac{1}{2}\frac{\partial^2 M_2}{\partial A_0^2}\beta^2\mu + \frac{\partial M_3}{\partial A_0}\beta\mu^2 + M_4\mu^3 + \ldots = 0 \tag{1.8}$$

Hereby it is assumed that $C_1(T_0)$ does not vanish identically. The values of the quantities M_n, for $n = 2, 3, 4$, are given* in the work [1].

In the case of multiple roots of the equation (1.7), the function $\beta(\mu)$, and hence also the periodic solutions $x(t)$, will be expressed in the form of series in integral as well as fractional powers of the parameter μ. We have

$$\beta = \sum_{n=1}^{\infty} A_{n/k}\mu^{n/k} \qquad (k = 1, 2, 3, \ldots) \tag{1.9}$$

In [1] there is considered the construction of the periodic solutions for a number of cases of double and triple roots of the equation (1.7). Hereby, the denominator of the fraction in the exponents of μ for double roots can take on the values $k = 1, 2$, while for triple roots the values of k are 1, 2, and 3.

2. Let us consider the stability of the periodic solutions of the equation (1.1) for double and triple roots of the equation of fundamental

* In [1], the term $-k^4 A_0 N_1^4/8$ is omitted in the formula for M_4.

amplitudes. The variational equation for the system (1.1) is

$$\frac{d^2y}{dt^2} - \mu \frac{\partial f}{\partial \dot{x}} \frac{dy}{dt} + \left(p^2 - \mu \frac{\partial f}{\partial x}\right) y = 0 \tag{2.1}$$

where $x(t)$ is some periodic solution of the equation (1.1).

The equation (2.1) is an equation with periodic coefficients. This equation has two particular solutions $y_1(t)$ and $y_2(t)$ which form a fundamental system [2] of solutions

$$\begin{aligned} y_1(t) &= \frac{\partial x(t)}{\partial A_0}, \qquad y_1(0) = 1, \qquad \dot{y}_1(0) = 0 \\ y_2(t) &= \dot{x}(t), \qquad y_2(0) = 0, \qquad \dot{y}_2(0) = \ddot{x}(0) \end{aligned} \tag{2.2}$$

The characteristic equation for the variational equation has the form

$$\rho^2 - 2A\rho + B = 0 \tag{2.3}$$

where

$$2A = y_1(T) + \frac{\dot{y}_2(T)}{\dot{y}_2(0)}, \qquad B = \frac{1}{\dot{y}_2(0)} [y_1(T)\dot{y}_2(T) - y_2(T)\dot{y}_1(T)] \tag{2.4}$$

Since the particular solution $y_2(t)$ of the variational equation is periodic, one of the roots of the characteristic equation (2.3) is $\rho_2 = 1$. Hence

$$\rho_1 = 2A - 1 = B$$

Note. In the investigation of the stability of the periodic solutions of non-autonomous systems, the sign of the quantity $2A - B - 1$ plays an important role [2]. For autonomous systems this quantity is zero.

If we substitute for the quantity $2A$ its expression from (2.4) and (2.2), we obtain

$$\rho_1 = \partial x(T) / \partial A_0 \tag{2.5}$$

According to the theorem of Andronov and Vitt, the periodic solutions of an autonomous system with one degree of freedom will be stable if

$$\rho_1 < 1 \tag{2.6}$$

As was mentioned above, for double and triple roots of the equation (1.7) the periodic solutions of the system (1.1) can be represented by series in powers of μ, $\mu^{1/2}$ and $\mu^{1/3}$. Let us compute the expansion $\partial x(T)/\partial A_0$ as power series in $\mu^{1/2}$ and $\mu^{1/3}$. The expansion in while powers of μ can be obtained as a particular case from the expansion in $\mu^{1/2}$.

We shall make use of the notation of the work [1], and write

$$P_n(A_1) = \frac{1}{n!}\frac{\partial^n C_1}{\partial A_0^{\ n}} A_1^{\ n} + \frac{1}{(n-1)!}\frac{\partial^{n-1} M_2}{\partial A_0^{\ n-1}} A_1^{\ n-1} + \ldots + M_{n+1}$$
$$Q_n(A_2) = \frac{1}{n!}\frac{\partial^n P_2}{\partial A_1^{\ n}} A_2^{\ n} + \frac{1}{(n-1)!}\frac{\partial^{n-1} P_3}{\partial A_1^{\ n-1}} A_2^{\ n-1} + \ldots + P_{n+2} \tag{2.7}$$

After laborious derivations we obtain the expansion in $\mu^{1/2}$,

$$\frac{\partial x(T)}{\partial A_0} = 1 + \frac{\partial C_1}{\partial A_0}\mu + \frac{\partial^2 C_1}{\partial A_0^{\ 2}} A_{1/2}\mu^{3/2} + \left(\frac{\partial P_2}{\partial A_1} + \frac{1}{2}\frac{\partial^3 C_1}{\partial A_0^{\ 3}} A_{1/2}^{\ \ 2}\right)\mu^2 + \left(\frac{\partial^2 C_1}{\partial A_0^{\ 2}} A_{3/2} + \ldots\right)\mu^{5/2} +$$
$$+ \left(\frac{\partial^2 C_1}{\partial A_0^{\ 2}} A_2 + \frac{\partial P_3}{\partial A_1} + \ldots\right)\mu^3 + \left(\frac{\partial^2 C_1}{\partial A_0^{\ 2}} A_{5/2} + \frac{\partial^2 P_2}{\partial A_1^{\ 2}} A_{3/2} + \ldots\right)\mu^{7/2} + \tag{2.8}$$
$$+ \left(\frac{\partial^2 C_1}{\partial A_0^{\ 2}} A_3 + \frac{\partial Q_3}{\partial A_2} + \frac{1}{2}\frac{\partial^3 C_1}{\partial A_0^{\ 3}} A_{3/2}^{\ \ 2} + \ldots\right)\mu^4 + \left(\frac{\partial^2 C_1}{\partial A_0^{\ 2}} A_{7/2} + \frac{\partial^2 P_3}{\partial A_1^{\ 2}} A_{5/2} + \ldots\right)\mu^{9/2} + \ldots$$

The omitted terms in the coefficients of the expansion (2.8) contain as factors $A_{1/2}$, and in the last coefficient $A_{1/2}$ and $A_{3/2}$.

Realizing that the expansion $\partial x(T)/\partial A_0$ as a series in $\mu^{1/3}$ is of interest only when there are triple roots (and roots of higher multiplicity) of the equation (1.7), we set

$$\frac{\partial C_1}{\partial A_0} = \frac{\partial^2 C_1}{\partial A_0^{\ 2}} = 0$$

We obtain

$$\frac{\partial x(T)}{\partial A_0} = 1 + \frac{1}{2}\frac{\partial^3 C_1}{\partial A_0^{\ 3}} A_{1/3}^{\ \ 2}\mu^{5/3} + \left(\frac{\partial M_2}{\partial A_0} + \ldots\right)\mu^2 + \left(\frac{1}{2}\frac{\partial^3 C_1}{\partial A_0^{\ 3}} A_{2/3}^{\ \ 2} + \ldots\right)\mu^{7/3} + \ldots +$$
$$+ \left(\frac{\partial P_2}{\partial A_1} + \ldots\right)\mu^3 + \ldots + \left(\frac{1}{2}\frac{\partial^3 C_1}{\partial A_0^{\ 3}} A_{4/3}^{\ \ 2} + \ldots\right)\mu^{11/3} + \left(\frac{\partial^2 P_2}{\partial A_1^{\ 2}} A_2 + \frac{\partial P_4}{\partial A_1} + \ldots\right)\mu^4 +$$
$$+ \left(\frac{1}{2}\frac{\partial^3 C_1}{\partial A_0^{\ 3}} A_{5/3}^{\ \ 2} + \ldots\right)\mu^{13/3} + \ldots \tag{2.9}$$

The omitted terms contain as factors $A_{n/3}$ with fractional indices which were already included in the preceding coefficients.

In what follows we shall assume that the parameter μ is so small that the signs of the sums of the series appearing in the right-hand parts of the formulas (2.8) and (2.9) after the number one, are determined by the first nonvanishing term.

All the cases considered in the article [1] are listed in the table, which shows for each case the form of the series (1.9), the coefficients with which these series start, and the corresponding conditions of stability. Therein the case 1 corresponds to simple roots, the case 2 to double roots, and the case 3 to triple roots of the equation of fundamental amplitudes (1.7). The various branches of the function $\beta(\mu)$ are denoted by $\beta^{(1)}$, $\beta^{(2)}$ and $\beta^{(3)}$.

We note that in the case of double roots, the equation (1.8), with $M_2 = 0$, can be transformed with the aid of the substitution $\beta = (A_1 + \gamma)\mu$ into the equation

$$\frac{\partial P_2}{\partial A_1}\gamma + P_3\mu + \frac{1}{2}\frac{\partial^2 C_1}{\partial A_0^2}\gamma^2 + \frac{\partial P_3}{\partial A_1}\gamma\mu + P_4\mu^2 + \ldots = 0 \qquad (2.10)$$

Hereby the coefficient A_1 is determined by means of a quadratic equation $P_2(A_1) = 0$.

Analogously, in case of triple roots, the equation (1.8) can be transformed, when $M_2 = 0$, $\partial M_2/\partial A_0 = 0$, and $M_3 = 0$, by the same substitution into

$$\frac{\partial P_3}{\partial A_1}\gamma + P_4\mu + \frac{1}{2}\frac{\partial^2 P_3}{\partial A_1^2}\gamma^2 + \frac{\partial P_4}{\partial A_1}\gamma\mu + P_5\mu^2 + \ldots = 0 \qquad (2.11)$$

The coefficient A_1 is determined in this case by means of a cubic equation $P_3(A_1) = 0$.

3. Let us first consider the characteristic cases when the equation (1.7) has double roots. Now the first coefficients of the expansion of the quantity β are determined from quadratic equations. Therefore there may exist either two solutions with real coefficients or none.

Let us take the case 2. 1, when $M_2 \neq 0$. In this case the coefficient $A_{1/2}$ is determined from the quadratic binomial equation [1]

$$\frac{1}{2}\frac{\partial^2 C_1}{\partial A_0^2}A_{1/2}^2 + M_2 = 0$$

If the roots of this equation are real then they can differ only in sign. The condition of stability takes the form

$$A_{1/2}\partial^2 C_1 / \partial A_0^2 < 0$$

If $M_2 = 0$, then $A_{1/2} = 0$, and the coefficient A_1 is determined from the quadratic equation $P_2(A_1) = 0$. As explained in the article [1], a solution in the form of a series in $\mu^{1/2}$ is obtained only in the case when this equation has multiple roots $A_1^{(1)} = A_1^{(2)}$. Then the coefficient $A_{3/2}$ will be determined from a quadratic binomial equation which is similar to the equation for $A_{1/2}$. The condition for stability in this case will be $A_{3/2}\partial^2 C_1/\partial A_0^2 < 0$. For the various branches of the function $\beta(\mu)$ we have the expansions

$$\beta^{(1)} = A_1\mu + A_{3/2}\mu^{3/2} + A_2^{(1)}\mu^2 + \ldots, \quad \beta^{(2)} = A_1\mu - A_{3/2}\mu^{3/2} + A_2^{(2)}\mu^2 + \ldots \qquad (3.1)$$

TABLE

Various cases	Coefficients of the equation (1.8)	k	Coefficients of the equation (1.9)	Conditions of stability
1	$M_2 \neq 0$	1	$A_1 \neq 0$	$\frac{\partial C_1}{\partial A_0} < 0$
2. 1	$M_2 \neq 0$	2	$A_{1/2} \neq 0$	$\frac{\partial^2 C_1}{\partial A_0^2} A_{1/2} < 0$
2. 2. *a)*	$M_2 = 0, \quad \frac{\partial P_2}{\partial A_1} \neq 0$	1	$A_{1/2} = 0, \quad A_1^{(1)} \neq A_1^{(2)}$	$\frac{\partial P_2}{\partial A_1} < 0$
b)	$\frac{\partial P_2}{\partial A_1} = 0, \quad P_3 \neq 0$	2	$A_1^{(1)} = A_1^{(2)}, \quad A_{3/2} \neq 0$	$\frac{\partial^2 C_1}{\partial A_0^2} A_{3/2} < 0$
c)	$P_3 = 0, \quad \frac{\partial Q_2}{\partial A_2} \neq 0$	1	$A_1^{(1)} = A_1^{(2)}, \quad A_2^{(1)} \neq A_2^{(2)}$	$\frac{\partial Q_2}{\partial A_2} < 0$
	$\frac{\partial Q_2}{\partial A_2} = 0, \quad Q_3 \neq 0$	2	$A_{3/2} = 0, \; A_2^{(1)} = A_2^{(2)}$ $A_{5/2} \neq 0$	$\frac{\partial^2 C_1}{\partial A_0^2} A_{5/2} < 0$
etc.				
3 . 1	$M_2 \neq 0$	3	$A_{1/3} \neq 0$	$\frac{\partial^3 C_1}{\partial A_0^3} < 0$
3 . 2	$M_2 = 0, \quad \frac{\partial M_2}{\partial A_0} \neq 0$	2	$A_{1/2} \neq 0$	$\frac{\partial^3 C_1}{\partial A_0^3} < 0$
		1	$A_{1/2} = 0, \quad A_1 \neq 0$	$\frac{\partial M_2}{\partial A_0} < 0$
3 . 3	$\frac{\partial M_2}{\partial A_0} = 0, \quad M_3 \neq 0$	3	$A_{1/3} = 0, \quad A_{2/3} \neq 0$	$\frac{\partial^3 C_1}{\partial A_0^3} < 0$
3 . 4	$M_3 = 0, \quad \frac{\partial P_3}{\partial A_1} \neq 0$	1	A_1 — various	$\frac{\partial P_3}{\partial A_1} < 0$
3 . 5. *a)*	$\frac{\partial P_3}{\partial A_1} = 0, \quad P_4 \neq 0$	2	$A_{1/2} = 0, \quad A_1^{(1)} = A_1^{(2)}$ $A_{3/2} \neq 0$	$\frac{\partial^2 P_3}{\partial A_1^2} A_{3/2} < 0$
b)	$P_4 = 0, \quad \frac{\partial Q_3}{\partial A_2} \neq 0$	1	$A_1^{(1)} = A_1^{(2)}, \quad A_2^{(1)} \neq A_2^{(2)}$	$\frac{\partial Q_3}{\partial A_2} < 0$
	$\frac{\partial Q_3}{\partial A_2} = 0, \quad Q_4 \neq 0$	2	$A_{3/2} = 0, \quad A_2^{(1)} = A_2^{(2)}$ $A_{5/2} \neq 0$	$\frac{\partial^2 P_3}{\partial A_1^2} A_{5/2} < 0$
etc.				
3 . 6. *a)*	$\frac{\partial^2 P_3}{\partial A_1^2} = 0, \quad P_4 \neq 0$	3	$A_1^{(1)} = A_1^{(2)} = A_1^{(3)}$ $A_{4/3} \neq 0$	$\frac{\partial^3 C_1}{\partial A_0^3} < 0$
b)	$P_4 = 0, \quad \frac{\partial P_4}{\partial A_1} \neq 0$	2	$A_1^{(1)} = A_1^{(2)} = A_1^{(3)}$ $A_{3/2} \neq 0$	$\frac{\partial^3 C_1}{\partial A_0^3} < 0$
		1	$A_{3/2} = 0, \quad A_2 \neq 0$	$\frac{\partial P_4}{\partial A_1} < 0$
c) etc.	$\frac{\partial P_4}{\partial A_1} = 0, \quad P_5 \neq 0$	3	$A_{4/3} = 0, \quad A_{5/3} \neq 0$	$\frac{\partial^3 C_1}{\partial A_0^3} < 0$

Analogously, if $A_{3/2} = 0$, then the expansion in terms of $\mu^{1/2}$ will exist only under the condition that $A_2^{(1)} = A_2^{(2)}$ and so on We note that if any one coefficient with a fractional index $A_{n/2} \neq 0$, then all successive coefficients are determined from linear equations.

Thus, if the equation (1.7) has a double root the solution is represented by a series in powers of $\mu^{1/2}$ if the first unequal coefficients $A_{n/2}$ have fractional indices. Hereby the condition of stability will have the form

$$A_{n/2}\partial^2 C_1 / \partial A_0^2 < 0 \tag{3.2}$$

where $A_{n/2}$ is the first non-zero coefficient with a fractional index. Since

$$A_{n/2}^{(1)} = - A_{n/2}^{(2)}$$

one of the solutions is stable, while the other one is unstable.

Now we consider the case 2. 2. *a*) when $M_2 = 0$ and hence, $A_{1/2} = 0$, while the quadratic equation $P_2(A_1) = 0$ has simple real roots. In this case the condition for stability is the inequality $\partial P_2/\partial A_1 < 0$. If, however, the roots of the equation $P_2(A_1) = 0$ happen to be multiple ones, and if $A_{3/2} = 0$, while the roots of the quadratic equation $Q_2(A_2) = 0$ are simple, then the condition for stability will be $\partial Q_2/\partial A_2 < 0$. Hereby,

$$\beta^{(1)} = A_1\mu + A_2^{(1)}\mu^2 + \ldots, \qquad \beta^{(2)} = A_1\mu + A_2^{(2)}\mu^2 + \ldots \tag{3.3}$$

The subsequent coefficients A_n will be determined from linear equations.

Thus, if in case of double roots of the equation (1.7), the first distinct coefficients $A_{n/2}$ appear with integer powers of μ, then the periodic solutions will be represented by series with integral powers of μ, and the condition of stability will have the form

$$\partial W_2 / \partial A_s < 0 \tag{3.4}$$

where $W_2(A_s) = 0$ is the first one of the quadratic equations for A_n ($n = 1, 2, \ldots$), which has multiple roots. Hereby one of the solutions will be stable while the other one will be unstable.

Next we consider the characteristic cases when the equation (1.7) has a triple root. Suppose that β can be expanded into a series in powers of $\mu^{1/3}$. An example of this case is 3. 1, when $M_2 \neq 0$. The coefficient $A_{1/3}$ is determined by the binomial cubic equation [1]

$$\frac{1}{6}\frac{\partial^3 C_1}{\partial A_0{}^3}A_{1/_3}{}^3 + M_2 = 0$$

Such equations have one real root and two imaginary roots. Hence, only one of the branches of $\beta(\mu)$ will be real. If $M_2 = 0$, then $A_{1/3} = 0$, and for the determination of $A_{2/3}$ we shall have an analogous binomial cubic equation $P_3(A_1) = 0$. The expansion of β in powers of $\mu^{1/3}$ can exist only when all roots of this equation are equal, i.e. when $\partial^2 P_3/\partial A_1{}^2 = 0$. The equation for the next following coefficient $A_{4/3}$ will also be a binomial cubic equation. If $A_{4/3} = A_{5/3} = 0$, then for the existence of an expansion in powers of $\mu^{1/3}$ it is necessary that the cubic equation $Q_4(A_2) = 0$ have all roots equal, and so on.

Thus, for the existence of a solution in the form of a power series in $\mu^{1/3}$ it is necessary that the first coefficients $A_{n/3}$, which are not equal to each other, have a fractional index. The condition of stability in this case will be

$$\partial^3 C_1 / \partial A_0{}^3 < 0 \tag{3.5}$$

Let us now consider the case in which two branches of the function $\beta(\mu)$ are represented by series in $\mu^{1/2}$, while one branch is given in integral powers of μ. As an example we may take the case 3. 2, when $M_2 = 0$, and $\partial M_2/\partial A_0 \neq 0$. For the determination of $A_{1/2}$ we have the equation

$$\left(\frac{1}{6}\frac{\partial^3 C_1}{\partial A_0{}^3}A_{1/_2}{}^2 + \frac{\partial M_2}{\partial A_0}\right)A_{1/_2} = 0$$

This equation determines two equal roots and one zero root. There will exist either three real branches of the function $\beta(\mu)$ or one branch. The condition will be

$$\frac{1}{2}\frac{\partial^3 C_1}{\partial A_0{}^3}A_{1/_2}{}^2 + \frac{\partial M_2}{\partial A_0} < 0$$

for the first two branches.

This condition reduces to (3.5) if one takes into account the equation for the coefficient $A_{1/2}$.

For the branch with $A_{1/2} = 0$, we have a different condition of stability

$$\partial M_2 / \partial A_0 < 0 \tag{3.6}$$

From the condition that $A_{1/2}$ be real it follows that the quantities which stand on the left-hand sides of the inequalities (3.5) and (3.6) have different signs. The branch of the function $\beta(\mu)$, with $A_{1/2} = 0$, which is representable by a series in integral powers of μ will lie between the other two branches. An analogous example is the case 3. 6. *b*), where

$$A_{1/2} = 0, \quad A_1^{(1)} = A_1^{(2)} = A_1^{(3)}$$

One can show that in this case the conditions of stability reduce to the inequalities

$$\partial^3 C_1 / \partial A_0^3 < 0, \qquad \partial P_4 / \partial A_1 < 0 \tag{3.7}$$

whereby the quantities on the left-hand sides of these inequalities have opposite signs. The first inequality applies to the branches of the function $\beta(\mu)$ which are representable as series in powers of $\mu^{1/2}$, while the second inequality is for the branch which can be represented as a series in integral powers of μ. This branch is located between the other two branches

$$\begin{aligned} \beta^{(1)} &= A_1\mu + A_{3/2}\mu^{3/2} + A_2^{(1)}\mu^2 + \ldots \\ \beta^{(2)} &= A_1\mu \qquad\qquad\;\; + A_2^{(2)}\mu^2 + \ldots \\ \beta^{(3)} &= A_1\mu - A_{3/2}\mu^{3/2} + A_2^{(3)}\mu^2 + \ldots \end{aligned} \tag{3.8}$$

In these and in similar cases, the solutions in the form of series in $\mu^{1/2}$ will be simultaneously stable or unstable, while the solutions in the form of series in integral powers of μ will have opposite natures in regard to stability.

An example, which is analogous to those considered above in the analysis of expansions in powers of $\mu^{1/2}$ for double roots of the equation (1.7), is the case 3. 5. *a*), Here the condition of stability, for the branches representable in series of powers of $\mu^{1/2}$, are

$$A_{3/2}\partial^2 P_3 / \partial A_1^2 < 0 \tag{3.9}$$

The second case 3. 5. *b*) is of the same nature.

Finally, there can occur cases in which all three branches of the function $\beta(\mu)$ are represented by series with integral powers of μ. Such an example is given by the case 3. 4, in which $M_2 = \partial M_2/\partial A_0 = M_3 = 0$, while the cubic equation $P_3(A_1) = 0$ has three simple real roots. The condition of stability will be

$$\partial P_3 / \partial A_1 < 0 \tag{3.10}$$

The stability and instability of the branches will, obviously, alternate.

A second example is the subcase of the case 3. 5. *b*). Here one of the roots of the equation $P_3(A_1) = 0$ is a double root, but $P_4 = 0$ and the equation $Q_3(A_2) = 0$ have simple roots. In this case

$$\begin{aligned} \beta^{(1)} &= A_1^{(1)}\mu + A_2^{(1)}\mu^2 + \ldots \\ \beta^{(2)} &= A_1^{(1)}\mu + A_2^{(2)}\mu^2 + \ldots \\ \beta^{(3)} &= A_1^{(3)}\mu + A_2^{(3)}\mu^2 + \ldots \end{aligned} \tag{3.11}$$

and to the condition of stability (3.10) for the third branch one must add the condition of stability for the two other branches

$$\partial Q_3 / \partial A_2 < 0 \tag{3.12}$$

We note that all equations, by means of which the first unequal coefficients A_n are determined, are nonlinear, while all succeeding euqations are linear.

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